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函子范畴的 incidence 代数的两个例子

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摘要: 以箭图 Q_1, Q_2, Q_3 为例, 构造有限偏序 k 范畴 Γ_1, Γ_2 , 考虑 Γ_1, Γ_2 及函子范畴 $\Gamma_1^{\Gamma_2}$ 诱导的 incidence 代数的两个例子。

关键词: 函子范畴; Incidence 代数; 同构

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Two examples of incidence algebra of the functor category

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Abstract: Taking the quivers Q_1, Q_2, Q_3 as an example, finite partially ordered k -category Γ_1, Γ_2 were constructed. Two examples of incidence algebra induced by Γ_1, Γ_2 and functor category $\Gamma_1^{\Gamma_2}$ were considered.

Keywords: functor category; incidence algebra; isomorphism

在范畴理论中, 函子是用来研究范畴之间的对应关系, 以两个范畴间的函子为对象, 函子间的自然变换为态射的范畴称为函子范畴。许多常见的范畴是函子范畴, 如常见的 G -集范畴是函子范畴^[1], 而且任意给定范畴可嵌入一个函子范畴。偏序集是联系代数、拓扑、逻辑等众多分支的一类重要数学对象, 偏序集在表示论发展中占据重要地位^[2]。incidence 代数是定义在有限偏序集上的一类代数, 多年来, incidence 代数一直是代数学研究领域的热点之一^[3]。本文探讨箭图诱导的 incident 代数, 得到关于 incident 代数维数的一个结果与一个猜想。

1 预备知识

为引用方便, 对重要的相关定义作简要的回顾。

定义 1 设 \mathcal{C} 是一个小范畴, \mathcal{D} 是范畴, 定义范畴 $\mathcal{D}^{\mathcal{C}}$ (称之为函子范畴): $\text{obj } \mathcal{D}^{\mathcal{C}} =$

$\{T | T: \mathcal{C} \rightarrow \mathcal{D} \text{ 为共变函子} \}$, $\text{Hom}_{\mathcal{D}^{\mathcal{C}}}(T, T') = \{ \tau: T \rightarrow T' \text{ 为自然变换} \}$, 合成是自然变换的合成。

定理 1^[4] 设 k 是有单位元 1 的交换环, 范畴 \mathcal{C} 为小范畴, 若范畴 \mathcal{D} 是 k 上小范畴, 则函子范畴 $\mathcal{D}^{\mathcal{C}}$ 仍为 k 上小范畴。

2 两个例子

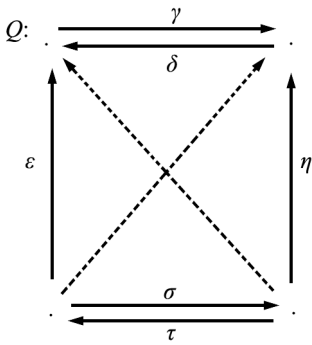
例 1 设 k 是有单位元 1 的交换环,

构造箭图 $Q_1: a \cdot \overset{\alpha}{\underset{\beta}{\rightleftarrows}} \cdot b, I = \langle \alpha\beta, \beta\alpha \rangle$

$\Leftrightarrow \alpha\beta = 0, \beta\alpha = 0$, 构造箭图 $Q_2: 1 \cdot \overset{l}{\longrightarrow} \cdot 2, kQ_1 = ke_a + ke_b + k\alpha + k\beta, kQ_2 = ke_1 + ke_2 + kl$, 记 $\Gamma_1 = kQ_1 - \text{mod}, \Gamma_2 = kQ_2 - \text{mod}$, 那么 Γ_1, Γ_2 是 k 上小范畴, 且同构于箭图 Q (图 1) 诱导的 10 维代数的模范畴, 其中, $\gamma\delta = 0, \delta\gamma = 0, \eta\tau = 0, \tau\eta = 0, \gamma\varepsilon = 0, \varepsilon\tau = 0, \eta\sigma = 0, \delta\eta = 0$ 。

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图1 Kronecker 箭图与 A_2 箭图诱导的 incidence 代数Fig.1 Incidence algebra induced by Kronecker quiver and A_2 quiver

证明:箭图 Q_1 可视为范畴 $\Gamma_1: \text{obj}: a, b, \alpha, \beta$, $\text{mor}: \text{Hom}(a, b) = \langle \alpha \rangle, \text{Hom}(b, a) = \langle \beta \rangle$, $\text{Hom}(a, a) = \langle e_a \rangle, \text{Hom}(b, b) = \langle e_b \rangle$.

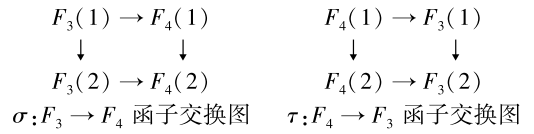
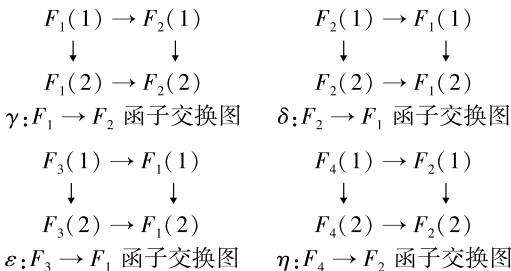
箭图 Q_2 可视为范畴 $\Gamma_2: \text{obj}: 1, 2$, $\text{mor}: \text{Hom}(1, 1) = \langle e_1 \rangle, \text{Hom}(2, 2) = \langle e_2 \rangle$, $\text{Hom}(1, 2) = \langle l \rangle$.

显然, Γ_1, Γ_2 都是 k 上小范畴. 根据定理 1, $\Gamma_1^{\Gamma_2}$ 也是 k 上小范畴: $\text{obj}: F_1, F_2, F_3, F_4$, $\text{mor}: \text{Hom}(F_1, F_1) = \langle ke_{F_1} \rangle, \text{Hom}(F_2, F_2) = \langle ke_{F_2} \rangle$, $\text{Hom}(F_3, F_3) = \langle ke_{F_3} \rangle, \text{Hom}(F_4, F_4) = \langle ke_{F_4} \rangle$, $\text{Hom}(F_1, F_2) = \langle k\gamma \rangle, \text{Hom}(F_2, F_1) = \langle k\delta \rangle$, $\text{Hom}(F_3, F_1) = \langle k\varepsilon \rangle, \text{Hom}(F_4, F_2) = \langle k\eta \rangle$, $\text{Hom}(F_3, F_4) = \langle k\sigma \rangle, \text{Hom}(F_2, F_1) = \langle k\tau \rangle$.

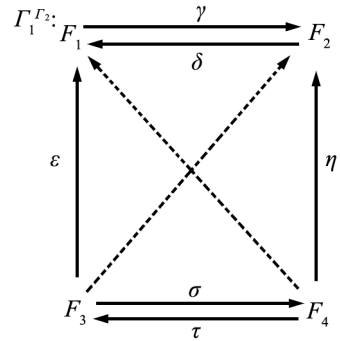
具体地, 有:

$\text{obj}: F_1: \Gamma_2 \rightarrow \Gamma_1: F_1(1) = a, F_1(2) = b, F_1(l) = \alpha, F_1(e_1) = e_a, F_1(e_2) = e_b, F_2: \Gamma_2 \rightarrow \Gamma_1: F_2(1) = b, F_2(2) = a, F_2(l) = \beta, F_2(e_1) = e_b, F_2(e_2) = e_a, F_3: \Gamma_2 \rightarrow \Gamma_1: F_3(1) = a, F_3(2) = a, F_3(l) = e_a, F_3(e_1) = e_a, F_3(e_2) = e_b, F_4: \Gamma_2 \rightarrow \Gamma_1: F_4(1) = b, F_4(2) = b, F_4(l) = e_b, F_4(e_1) = e_a, F_4(e_2) = e_b$.

$\text{mor}: e_{F_1}: F_1 \rightarrow F_1, e_{F_2}: F_2 \rightarrow F_2, e_{F_3}: F_3 \rightarrow F_3, e_{F_4}: F_4 \rightarrow F_4, \gamma = (\gamma_1, \gamma_2): F_1 \rightarrow F_2, \delta = (\delta_1, \delta_2): F_2 \rightarrow F_1, \varepsilon = (\varepsilon_1, \varepsilon_2): F_3 \rightarrow F_1, \eta = (\eta_1, \eta_2): F_4 \rightarrow F_2, \sigma = (\sigma_1, \sigma_2): F_3 \rightarrow F_4, \tau = (\tau_1, \tau_2): F_4 \rightarrow F_3$ 满足函子交换图(图 2)如下。

图2 模 $\gamma, \delta, \varepsilon, \eta, \sigma, \tau$ 的函子交换图Fig. 2 Functor exchange graph of $\text{mor } \gamma, \delta, \varepsilon, \eta, \sigma, \tau$

$k\Gamma_1^{\Gamma_2} = ke_{F_1} + ke_{F_2} + ke_{F_3} + ke_{F_4} + k\gamma + k\delta + k\varepsilon + k\eta + k\sigma + k\tau$, 故 $\Gamma_1^{\Gamma_2}$ (图 3) 同构于 Γ 诱导的 10 维代数的模范畴, 其中 $\gamma\delta = 0, \delta\gamma = 0, \eta\tau = 0, \tau\eta = 0, \gamma\varepsilon = 0, \varepsilon\tau = 0, \eta\sigma = 0, \delta\eta = 0$.

图3 $\Gamma_1^{\Gamma_2}$ 范畴中对象关系图Fig. 3 Objects' relation graph in category $\Gamma_1^{\Gamma_2}$

例 2 设 k 是有单位元 1 的交换环,

构造箭图 $Q_1: a \cdot \xrightleftharpoons[\beta]{\alpha} \cdot b, I = \langle \alpha\beta, \beta\alpha \rangle$

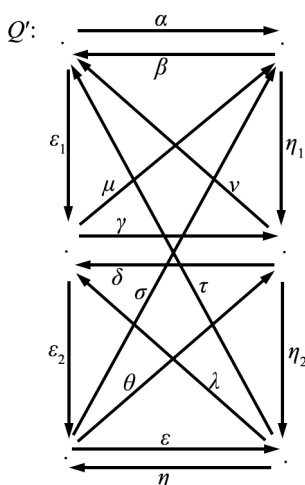
$\Leftrightarrow \alpha\beta = 0, \beta\alpha = 0$,

构造箭图 $Q_3: 1 \cdot \xrightarrow{l} \cdot 2 \xrightarrow{m} \cdot 3, ml \neq 0$,

$kQ_1 = ke_a + ke_b + k\alpha + k\beta, kQ_3 = ke_1 + ke_2 + ke_3 + kl + km + k(ml)$, 记 $\Gamma_1 = kQ_1 - \text{mod}, \Gamma_3 = kQ_3 - \text{mod}$, 那么 $\Gamma_1^{\Gamma_3}$ 是 k 上小范畴, 且同构于箭图 Q' (图 4) 诱导的 24 维代数的模范畴, 其中, $\alpha\beta = 0, \beta\alpha = 0, \gamma\delta = 0, \delta\gamma = 0, \varepsilon\eta = 0, \eta\varepsilon = 0, \theta\eta = 0, \delta\theta = 0, \mu\delta = 0, \beta\mu = 0, \lambda\varepsilon = 0, \gamma\lambda = 0, \nu\gamma = 0, \alpha\nu = 0$, 且 $\mu\varepsilon_1 = \alpha, \nu\eta_1 = \beta, \theta\varepsilon_2 = \gamma, \lambda\eta_2 = \delta, \varepsilon_2\lambda = \eta, \sigma\varepsilon_2 = \mu, \tau\eta_2 = \nu$.

证明: 类似例 1, 箭图 Q_3 可视为范畴 $\Gamma_3: \text{obj}: 1, 2, 3, m, l, ml$, $\text{mor}: \text{Hom}(1, 1) = \langle e_1 \rangle, \text{Hom}(2, 2) = \langle e_2 \rangle, \text{Hom}(3, 3) = \langle e_3 \rangle, \text{Hom}(1, 2) = \langle m \rangle, \text{Hom}(2, 3) = \langle l \rangle, \text{Hom}(1, 3) = \langle ml \rangle$.

与例 1 同理可得 $\Gamma_1^{\Gamma_3}$ 也是 k 上小范畴: $\text{obj}: F_1, F_2, F_3, F_4, F_5, F_6$.

图 4 Kronecker 箭图与 A_3 箭图诱导的 incidence 代数Fig.4 Incidence algebra induced by Kronecker quiver and A_3 quiver

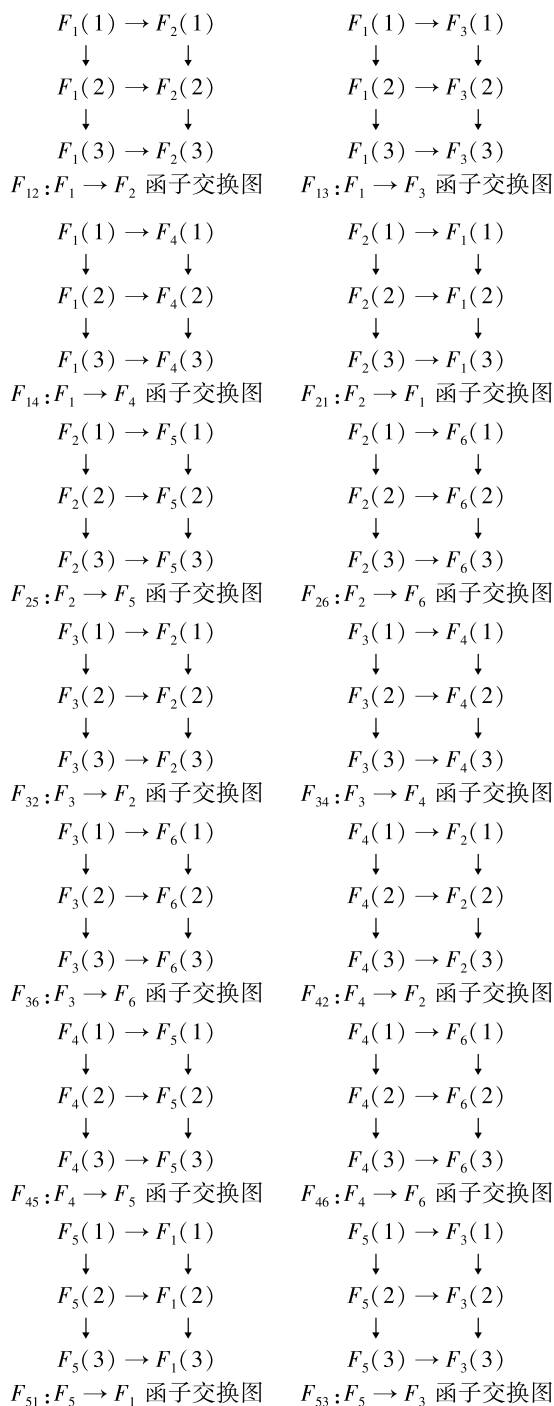
$\text{mor: Hom}(F_1, F_1) = \langle F_{11} \rangle, \text{Hom}(F_2, F_2) = \langle F_{22} \rangle, \text{Hom}(F_3, F_3) = \langle F_{33} \rangle, \text{Hom}(F_4, F_4) = \langle F_{44} \rangle, \text{Hom}(F_5, F_5) = \langle F_{55} \rangle, \text{Hom}(F_6, F_6) = \langle F_{66} \rangle,$
 $\text{Hom}(F_1, F_2) = \langle F_{12} \rangle, \text{Hom}(F_1, F_3) = \langle F_{13} \rangle, \text{Hom}(F_1, F_4) = \langle F_{14} \rangle, \text{Hom}(F_2, F_1) = \langle F_{21} \rangle,$
 $\text{Hom}(F_2, F_5) = \langle F_{25} \rangle, \text{Hom}(F_2, F_6) = \langle F_{26} \rangle, \text{Hom}(F_3, F_2) = \langle F_{32} \rangle, \text{Hom}(F_3, F_4) = \langle F_{34} \rangle,$
 $\text{Hom}(F_3, F_6) = \langle F_{36} \rangle, \text{Hom}(F_4, F_2) = \langle F_{42} \rangle, \text{Hom}(F_4, F_5) = \langle F_{45} \rangle, \text{Hom}(F_4, F_6) = \langle F_{46} \rangle,$
 $\text{Hom}(F_5, F_1) = \langle F_{51} \rangle, \text{Hom}(F_5, F_3) = \langle F_{53} \rangle, \text{Hom}(F_5, F_4) = \langle F_{54} \rangle, \text{Hom}(F_6, F_1) = \langle F_{61} \rangle,$
 $\text{Hom}(F_6, F_3) = \langle F_{63} \rangle, \text{Hom}(F_6, F_5) = \langle F_{65} \rangle.$

具体地,有:

$\text{obj: } F_1: \Gamma_3 \rightarrow \Gamma_1: F_1(1) = a, F_1(2) = a, F_1(3) = a, F_1(l) = e_a, F_1(m) = e_a, F_1(ml) = e_a, F_1(e_1) = e_a,$
 $F_1(e_2) = e_a, F_1(e_3) = e_a; F_2: \Gamma_3 \rightarrow \Gamma_1: F_2(1) = b, F_2(2) = b, F_2(3) = b, F_2(l) = e_b, F_2(m) = e_b,$
 $F_2(ml) = e_b, F_2(e_1) = e_b, F_2(e_2) = e_b, F_2(e_3) = e_b;$
 $F_3: \Gamma_3 \rightarrow \Gamma_1: F_3(1) = a, F_3(2) = a, F_3(3) = b, F_3(l) = e_a, F_3(m) = \alpha, F_3(ml) = \alpha, F_3(e_1) = e_a,$
 $F_3(e_2) = e_a, F_3(e_3) = e_b; F_4: \Gamma_3 \rightarrow \Gamma_1: F_4(1) = a, F_4(2) = b, F_4(3) = b, F_4(l) = \alpha, F_4(m) = e_b,$
 $F_4(ml) = \alpha, F_4(e_1) = e_a, F_4(e_2) = e_b, F_4(e_3) = e_b;$
 $F_5: \Gamma_3 \rightarrow \Gamma_1: F_5(1) = b, F_5(2) = a, F_5(3) = a, F_5(l) = \beta, F_5(m) = e_a, F_5(ml) = \beta, F_5(e_1) = e_b,$
 $F_5(e_2) = e_a, F_5(e_3) = e_a; F_6: \Gamma_3 \rightarrow \Gamma_1: F_6(1) = b, F_6(2) = b, F_6(3) = a, F_6(l) = e_b, F_6(m) = \beta,$

$F_6(ml) = \beta, F_6(e_1) = e_b, F_6(e_2) = e_b, F_6(e_3) = e_a.$

$\text{mor: } F_{11}: F_1 \rightarrow F_1, F_{22}: F_2 \rightarrow F_2, F_{33}: F_3 \rightarrow F_3,$
 $F_{44}: F_4 \rightarrow F_4, F_{55}: F_5 \rightarrow F_5, F_{66}: F_6 \rightarrow F_6, F_{12}: F_1 \rightarrow F_2,$
 $F_{13}: F_1 \rightarrow F_3, F_{14}: F_1 \rightarrow F_4, F_{21}: F_2 \rightarrow F_1,$
 $F_{25}: F_2 \rightarrow F_5, F_{26}: F_2 \rightarrow F_6, F_{32}: F_3 \rightarrow F_2, F_{34}: F_3 \rightarrow F_4,$
 $F_{36}: F_3 \rightarrow F_6, F_{42}: F_4 \rightarrow F_2, F_{45}: F_4 \rightarrow F_5,$
 $F_{46}: F_4 \rightarrow F_6, F_{51}: F_5 \rightarrow F_1, F_{53}: F_5 \rightarrow F_3, F_{54}: F_5 \rightarrow F_4,$
 $F_{61}: F_6 \rightarrow F_1, F_{63}: F_6 \rightarrow F_3, F_{65}: F_6 \rightarrow F_5$
 满足函子交换图(图 5)如下。



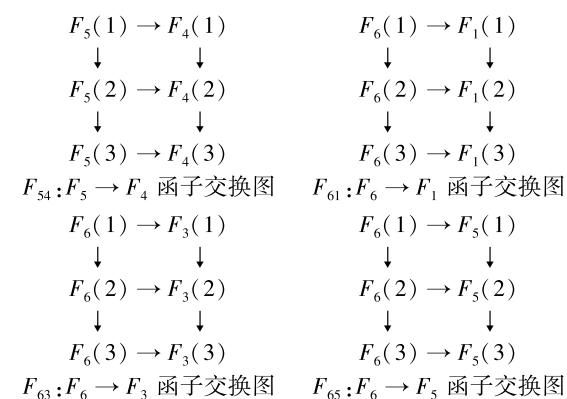


图 5 $F_{ij}: F_i \rightarrow F_j (i=1, \cdots, 6,)$ 的函子交换图

Fig. 5 Functor exchange graph of mor
 $F_{ij}: F_i \rightarrow F_j (i=1, \cdots, 6,)$

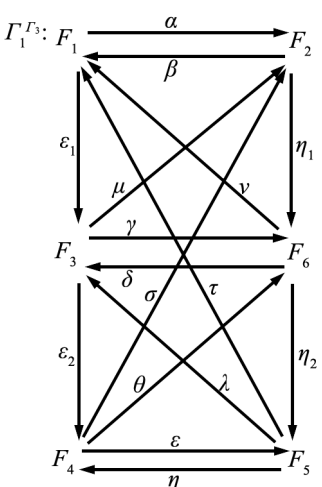


图 6 $\Gamma_1^{\Gamma_3}$ 中对象关系图

Fig. 6 Objects' relation graph in category $\Gamma_1^{\Gamma_3}$

故 $\Gamma_1^{\Gamma_3}$ (图 6) 同构于 Q' 诱导的 24 维代数的模范畴。其中, $\alpha\beta = 0, \beta\alpha = 0, \gamma\delta = 0, \delta\gamma = 0, \varepsilon\eta = 0, \eta\varepsilon = 0, \theta\eta = 0, \delta\theta = 0, \mu\delta = 0, \beta\mu = 0, \lambda\varepsilon = 0, \gamma\lambda = 0, \nu\gamma = 0, \alpha\nu = 0$ 。

注 1: 若 $\mathcal{P}_1, \mathcal{P}_2$ 均为有限偏序 k -范畴, 记由范畴 $\mathcal{P}_1, \mathcal{P}_2$, 函子范畴 $\mathcal{P}_1^{\mathcal{P}_2}$ 诱导的 incidence 代数分别为 A, B, C , $\mathcal{P}_1, \mathcal{P}_2$ 的点数分别记为 $|\mathcal{P}_1|, |\mathcal{P}_2|$ 。

根据例 1 得到: $\dim_k C = \dim_k A \times |\mathcal{P}_2| + |\mathcal{P}_1| \times (\dim_k B - |\mathcal{P}_2|)$, 而根据例 2 有: $\dim_k C = 24 \neq \dim_k A \times |\mathcal{P}_2| + |\mathcal{P}_1| \times (\dim_k B - |\mathcal{P}_2|) = 4 \times 3 + 2 \times (6 - 3) = 18$ 。

注 2: 设 $Q_1 = A_2$ 箭图, $Q_2 = A_n$ 箭图, 令 $A = kQ_1, B = kQ_2, C = k(Q_1^{Q_2})$, 则有: $\dim_k C = \dim_k A \times |\mathcal{P}_1| + |\mathcal{P}_1| \times (\dim_k B + 1 - |\mathcal{P}_2|)$ 。由此, 我们大胆猜想以上结论对其它情况可能也成立, 有待进一步考究证明。

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